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Existence of solutions for nonlinear impulsive higher order fractional differential equations*

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Abstract. In this paper, we prove the existence of solutions for nonlinear impulsive differential equations of fractional order $q \in (n-1, n]$. The results are based on Altman's fixed point theorem and Leray-Schauder's fixed point theorem.

Keywords: Nonlinear fractional differential equations; Impulse; Initial value problem; Existence of Solutions; Fixed point theorem.

1 Introduction

In this paper, we consider the existence and uniqueness of solutions of the following impulsive problems:

$$\begin{cases} {}^{C}D_{t_i}^{q}u(t) = f(t, u(t)), n-1 < q \leq n, t \in J_i = (t_i, t_{i+1}], \\ \triangle u^{(j)}(t_k) = I_{j,k}(u(t_k))(j = 0, 1, \ldots, n-1), k = 1, 2, \ldots, p, \\ u^{(j)}(0) = u_j(j = 0, 1, \ldots, n-1), \end{cases} \tag{1}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative, $f \in C(J \times R, R)$, $I_{j,k} \in C(R, R)$ $(j = 0, 1, \ldots, n-1)$, $J = [0, 1]$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = 1$, $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$,

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where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limit of $u(t)$ at $t = t_k(k = 1, 2, \dots, p)$, respectively. $\triangle u^{(j)}(t_k)$ $(j = 1, 2, \dots, n-1)$ have a similar meaning for $u^{(j)}(t)$ $(j = 1, 2, \dots, n-1)$.

Recently, the subject of fractional differential equations has emerged as an important area of investigation. Indeed, we can find numerous applications in electrochemistry, control, electromagnetic, porous media, etc.[1-4]. Therefore, they have received much attention. For the most recent works for the existence and uniqueness of solutions of the initial and boundary value problems for nonlinear fractional differential equations, we mention [5-16, 25-30]. But, as far as we know, there have been few papers which have considered the multi-order fractional differential equations can be found in [18-24].

The organization of this paper is as follows: In Section 2, we give some basic definitions and properties. In Section 3, will be devoted to existence and uniqueness results for nonlinear impulsive differential equations of fractional order.

2 Preliminaries

Let us set $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots, J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, 1]$ and introduce the spaces:
$PC(J, R) = \{u : J \to R \mid u \in C(J_k), k = 0, 1, \dots, p, \text{ and } u(t_k^+) \text{ exist } , k = 1, 2, \dots, p\}$ with the norm $\|u\| = \sup_{t \in J} |u(t)|$, and
$PC^{n-1}(J, R) = \{u : J \to R \mid u \in C^{n-1}(J_k), k = 0, 1, \dots, p, \text{ and } u^{(j)}(t_k^+)$
exist $, j = 0, 1, \dots, n-1, k = 1, 2, \dots, p\}$,
with the norm $\|u\|_{PC^{n-1}} = \max\{\|u\|, \|u'\|, \dots, \|u^{(n-1)}\|\}$. Obviously, $PC(J, R)$ and $PC^{n-1}(J, R)$ are Banach spaces.

Definition 2.1. A function $u \in PC^{n-1}(J, R)$ with its Caputo derivative of order q existing on J is a solution of (1) if it satisfies (1).

Theorem 2.1 ([17]). Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that $\|Tu\| \leq \|u\|, \forall u \in \partial\Omega$. Then T has a fixed point in $\overline{\Omega}$.

Theorem 2.2 ([17]). Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set $V = \{u \in E \mid u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E.

Lemma 2.1 ([1,9]). For $q > 0$, the general solution of the fractional differential equation ${}^C D^q u(t) = 0$ is given by
$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, C_i \in R, i = 0, 1, \dots, n-1\ , n = [q] + 1,$

where $[q]$ denotes the integer part of the real number q.

In view of Lemma 2.1, it follows that

$$I^{q\,C}D^{q}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}, \tag{2}$$

for some $C_i \in R, i = 0, 1, \ldots, n-1$, $n = [q] + 1$.

Lemma 2.2. For a given $y \in C[0,1]$, the unique solution of the following impulsive boundary value problem

$$\begin{cases} {}^{C}D^{q}_{t_i}u(t) = y(t), n-1 < q \le n, t \in J_i = (t_i, t_{i+1}], \\ \triangle u^{(j)}(t_k) = I_{j,k}(u(t_k))(j = 0, 1, \ldots, n-1), k = 1, 2, \ldots, p, \\ u^{(j)}(0) = u_j(j = 0, 1, \ldots, n-1), \end{cases} \tag{3}$$

is given by

$$u(t) = \begin{cases} \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}y(s)ds + \sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j+1)}t^j, \quad t \in J_0; \\ \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_k}^t (t-s)^{q-1}y(s)ds + \sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-j-1}y(s)ds \\ +\displaystyle\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-j-1}y(s)ds + \sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)}I_{j,k}(u(t_k)) \\ +\displaystyle\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)}I_{j,i}(u(t_i)) + \sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j+1)}t^j, \quad t \in J_k,\ k = 1, 2, \ldots, p. \end{cases} \tag{4}$$

Proof. Let u be a solution of (3). Then, by (2), we have

$$u(t) = I^q y(t) - \sum_{j=0}^{n-1} c_{j+1}t^j = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds - \sum_{j=0}^{n-1}c_{j+1}t^j, \quad t \in J_0, \tag{5}$$

for some $c_1, c_2, \ldots, c_n \in R$. Furthermore

$$u'(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)ds - c_2 - 2c_3t - 3c_4t^2 - \cdots - (n-1)c_nt^{n-2}, \quad t \in J_0,$$

$$u''(t) = \frac{1}{\Gamma(q-2)}\int_0^t (t-s)^{q-3}y(s)ds - 2c_3 - 6c_4t - \cdots - (n-1)(n-2)c_nt^{n-3}, \quad t \in J_0,$$

$$\cdots$$

$$u^{(n-1)}(t) = \frac{1}{\Gamma(q-n+1)}\int_0^t (t-s)^{q-n}y(s)ds - (n-1)!c_n, \quad t \in J_0,$$

if $t \in J_1$, then

$$u(t) = \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}y(s)ds - \sum_{j=0}^{n-1}d_{j+1}(t-t_1)^j,$$

$$u'(t)=\frac{1}{\Gamma(q-1)}\int_{t_1}^{t}(t-s)^{q-2}y(s)ds-d_2-2d_3(t-t_1)-3d_4(t-t_1)^2-\cdots-(n-1)d_n(t-t_1)^{n-2},$$

$$u''(t)=\frac{1}{\Gamma(q-2)}\int_{t_1}^{t}(t-s)^{q-3}y(s)ds-2d_3-6d_4(t-t_1)-\cdots-(n-1)(n-2)d_n(t-t_1)^{n-3},$$

$$\cdots$$

$$u^{(n-1)}(t)=\frac{1}{\Gamma(q-n+1)}\int_{t_1}^{t}(t-s)^{q-n}y(s)ds-(n-1)!d_n,$$

for some $d_1, d_2, \ldots, d_n \in R$. Thus we have

$$u(t_1^-)=\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds-c_1-c_2t_1-c_3t_1^2-c_4t_1^3-\cdots-c_nt_1^{n-1},\quad u(t_1^+)=-d_1,$$

$$u'(t_1^-)=\frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds-c_2-2c_3t_1-3c_4t_1^2-\cdots-(n-1)c_nt_1^{n-2}, u'(t_1^+)=-d_2,$$

$$u''(t_1^-)=\frac{1}{\Gamma(q-2)}\int_0^{t_1}(t_1-s)^{q-3}y(s)ds-2c_3-6c_4t_1-\cdots-(n-1)(n-2)c_nt_1^{n-3}, u''(t_1^+)=-2d_3,$$

$$\cdots$$

$$u^{(n-1)}(t_1^-)=\frac{1}{\Gamma(q-n+1)}\int_0^{t_1}(t_1-s)^{q-n}y(s)ds-(n-1)!c_n, u^{(n-1)}(t_1^+)=-(n-1)!d_n,$$

In view of $\triangle u(t_1)=u(t_1^+)-u(t_1^-)=I_{0,1}(u(t_1))$, $\triangle u'(t_1)=u'(t_1^+)-u'(t_1^-)=I_{1,1}(u(t_1))$, $\triangle u''(t_1)=u''(t_1^+)-u''(t_1^-)=I_{2,1}(u(t_1))$, …, $\triangle u^{(n-1)}(t_1)=u^{(n-1)}(t_1^+)-u^{(n-1)}(t_1^-)=I_{n-1,1}(u(t_1))$, we have

$$-d_1=\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds-c_1-c_2t_1-c_3t_1^2-c_4t_1^3-\cdots-c_nt_1^{n-1}+I_{0,1}(u(t_1)),$$

$$-d_2=\frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds-c_2-2c_3t_1-3c_4t_1^2-\cdots-(n-1)c_nt_1^{n-2}+I_{1,1}(u(t_1)),$$

$$-2d_3=\frac{1}{\Gamma(q-2)}\int_0^{t_1}(t_1-s)^{q-3}y(s)ds-2c_3-6c_4t_1-\cdots-(n-1)(n-2)c_nt_1^{n-3}+I_{2,1}(u(t_1)),$$

$$\cdots$$

$$-(n-1)!d_n=\frac{1}{\Gamma(q-n+1)}\int_0^{t_1}(t_1-s)^{q-n}ds-(n-1)!c_n+I_{n-1,1}(u(t_1)).$$

Consequently

$$\begin{aligned}u(t) \quad = \quad & \frac{1}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}y(s)ds+\sum_{j=0}^{n-1}\frac{(t-t_1)^j}{\Gamma(j+1)\Gamma(q-j)}\int_0^{t_1}(t_1-s)^{q-j-1}y(s)ds\\ &+\sum_{j=0}^{n-1}\frac{(t-t_1)^j}{\Gamma(j+1)}I_{j,1}(u(t_1))-\sum_{j=0}^{n-1}c_{j+1}t^j,\quad t\in J_1.\end{aligned}$$

Similarly, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}y(s)ds+\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-j-1}y(s)ds \\ &\quad +\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-j-1}y(s)ds+\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)}I_{j,k}(u(t_k)) \\ &\quad +\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)}I_{j,i}(u(t_i))-\sum_{j=0}^{n-1}c_{j+1}t^j, t\in J_k,\ k=1,2,\ldots,p. \end{aligned} \tag{6}$$

By $u^{(j)}(0)=u_j$, we have

$$c_{j+1}=\frac{-u_j}{\Gamma(j+1)}.$$

It follows that the solution given by (4) satisfies (3). This completes the proof.

3 Main Results

Define an operator $T: PC(J,R)\to PC(J,R)$ as

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f(s,u(s))ds \\ &\quad +\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-j-1}f(s,u(s))ds \\ &\quad +\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-j-1}f(s,u(s))ds \\ &\quad +\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j+1)}I_{j,k}(u(t_k))+\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)}I_{j,i}(u(t_i)) \\ &\quad +\sum_{j=0}^{n-1}m_{j+1}t^j,\quad t\in J_k,\ k=1,2,\ldots,p. \end{aligned} \tag{7}$$

where $m_{j+1}=-c_{j+1}(j=0,1,\ldots,n-1)$. Using Lemma 2.2 with $y(t)=f(t,u(t))$, problem (1) reduces to a fixed point problem $u=Tu$, where T is given by (7). Thus problem (1) has a solution if the operator T has a fixed point.

Theorem 3.1. Let $\lim_{\mathrm{u}\to 0}\frac{\mathrm{f(t,u)}}{\mathrm{u}}=0$, $\lim_{\mathrm{u}\to 0}\frac{\mathrm{I_{j,k}(u)}}{\mathrm{u}}=0$, $(j=0,1,\ldots,n-1)$ then problem (1) has at least one solution.

Proof. Firstly, we show that the operator $T: PC(J,R)\to PC(J,R)$ is completely continuous. Note that T is continuous in view of continuity of f and $I_{j,k}$. Let $\Omega\subset PC(J,R)$ be bounded. Then, there exist positive constants $L_i > 0(i = 1, 2, \ldots, n+1)$ such that $|f(t, u)| \leq L_1$, $|I_{j,k}(u)| \leq L_{j+2}$,$(j = 0, 1, \ldots, n-1)$, $\forall u \in \Omega$. Thus, $\forall u \in \Omega$, we have

$$|m_j| \leq \frac{u_j}{\Gamma(j+1)}.$$

Therefore

$$\begin{aligned}
|Tu(t)| &= \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} |f(s, u(s))| ds \\
&\quad + \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-j-1} |f(s, u(s))| ds \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-j-1} |f(s, u(s))| ds \\
&\quad + \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)} |I_{j,k}(u(t_k))| + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)} |I_{j,i}(u(t_i))| + \sum_{j=0}^{n-1} |m_{j+1}| \\
&\leq \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{(t-t_k)^j L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j L_1}{\Gamma(j+1)} \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{(t-t_k)^j L_j}{\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{u_j}{\Gamma(j+1)} \\
&\leq \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\
&\quad + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{u_j}{\Gamma(j+1)},
\end{aligned} \tag{8}$$

which implies that

$$\begin{aligned}
\|Tu\| &\leq \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\
&\quad + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{u_j}{\Gamma(j+1)} := L.
\end{aligned}$$

On the other hand, for any $t \in J_k$, $0 \leq k \leq p$, we have

$$\begin{aligned}
|Tu'(t)| &= \frac{1}{\Gamma(q-1)} \int_{t_k}^{t} (t-s)^{q-2} |f(s, u(s))| ds \\
&\quad + \sum_{j=0}^{n-1} \frac{(t-t_k)^{j-1}}{\Gamma(j)\Gamma(q-j)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-j-1} |f(s, u(s))| ds \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^{j-1}}{\Gamma(j)\Gamma(q-j)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-j-1} |f(s, u(s))| ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{\Gamma(j)}|I_{j,k}(u(t_k))|+\sum_{j=0}^{n-1}\sum_{i=1}^{k-1}\frac{[(t-t_k)+(t_k-t_i)]^{j-1}}{\Gamma(j)}|I_{j,i}(u(t_i))|+\sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j)}\\
\leq\ &\frac{L_1}{\Gamma(q)}+\sum_{j=0}^{n-1}\frac{L_1}{\Gamma(j)\Gamma(q-j+1)}+\sum_{j=0}^{n-1}\frac{(p-1)L_1}{\Gamma(j)}+\sum_{j=0}^{n-1}\frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j)}\\
&+\sum_{j=0}^{n-1}\frac{L_{j+2}}{\Gamma(j)}+\sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j)}:=\overline{L}.
\end{aligned}
$$

Hence, for $t_1, t_2 \in J_k$ with $t_1 < t_2$, $0 \le k \le p$, we have

$$
|(Tu)(t_2)-(Tu)(t_1)| \le \int_{t_1}^{t_2} |(Tu)'(s)|ds \le \overline{L}(t_2-t_1).
$$

This implies that T is equicontinuous on all the subintervals J_k, $k = 0, 1, \dots, p$. Thus, by Arzela-Ascoli Theorem, it follows that $T : PC(J, R) \to PC(J, R)$ is completely continuous. Now, in view of $\lim_{u\to 0}\frac{f(t,u)}{u} = 0$, $\lim_{u\to 0}\frac{I_{j,k}(u)}{u} = 0$ ($j = 0, 1, \dots, n-1$), there exists a constant $r > 0$ such that $|f(t,u)| \le \delta_1|u|$, $|I_{j,k}(u)| \le \delta_{j+2}|u|$ ($j = 0, 1, \dots, n-1$) for $0 < |u| < r$, where $\delta_i > 0 (i = 1, 2, \dots, n+1)$ satisfy

$$
\begin{aligned}
&\frac{\delta_1}{\Gamma(q+1)}+\sum_{j=0}^{n-1}\frac{\delta_{j+2}}{\Gamma(j+1)\Gamma(q-j+1)}+\sum_{j=0}^{n-1}\frac{(p-1)\delta_{j+2}}{\Gamma(j+1)}+\sum_{j=0}^{n-1}\frac{(p-1)\delta_1}{\Gamma(q-j+1)\Gamma(j+1)}\\
&+\sum_{j=0}^{n-1}\frac{\delta_{j+2}}{\Gamma(j+1)}+\sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j+1)}\le 1. \qquad (9)
\end{aligned}
$$

Define $\Omega = \{u \in PC(J, R) \mid \|u\| < r\}$ and take $u \in PC(J, R)$ such that $\|u\| = r$ so that $u \in \partial\Omega$. Then, by the process used to obtain (8), we have

$$
\begin{aligned}
|Tu(t)| \quad \le\quad &\{\frac{L_1}{\Gamma(q+1)}+\sum_{j=0}^{n-1}\frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)}+\sum_{j=0}^{n-1}\frac{(p-1)L_1}{\Gamma(j+1)}\\
&+\sum_{j=0}^{n-1}\frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)}+\sum_{j=0}^{n-1}\frac{L_j}{\Gamma(j+1)}+\sum_{j=0}^{n-1}\frac{u_j}{\Gamma(j+1)}\}\|u\|. \qquad (10)
\end{aligned}
$$

which implies that $\|Tu\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 2.1, the operator T has at least one fixed point, which implies that (1) has at least one solution $u \in \overline{\Omega}$.

Theorem 3.2. Assume that

(H_1) there exist positive constants $L_i (i = 1, 2, \dots, n+1)$ such that $|f(t,u)| \le L_1$, $|I_{j,k}(u)| \le L_{j+2}$,$(j = 0, 1, \dots, n-1)$ for $t \in J$, $u \in R$ and $k = 1, 2, \dots, p$. Then problem (1) has at least one solution.

Proof. As shown in Theorem 3.1, the operator $T : PC(J, R) \to PC(J, R)$ is completely continuous. Now, we show the set $V = \{u \in PC(J, R) | u = \mu Tu, 0 < \mu < 1\}$ is bounded. Let $u \in V$, then

$u = \mu A u, 0 < \mu < 1$. For any $t \in J$, we have

$$\begin{aligned} u(t) &= \frac{\mu}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s,u(s))ds \\ &+ \sum_{j=0}^{n-1} \frac{\mu(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-j-1} f(s,u(s))ds \\ &+ \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{\mu[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{i-1}}^{t_i} (t_i-s)^{q-j-1} f(s,u(s))ds \\ &+ \sum_{j=0}^{n-1} \frac{\mu(t-t_k)^j}{\Gamma(j+1)} I_{j,k}(u(t_k)) + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{\mu[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)} I_{j,i}(u(t_i)) \\ &+ \sum_{j=0}^{n-1} \mu m_{j+1} t^j, \end{aligned} \tag{11}$$

where $m_{j+1} = -c_{j+1} (j = 0, 1, \ldots, n-1)$. Combining (H_1) and (11), we obtain

$$\begin{aligned} |u(t)| &= \mu |Tu(t)| \\ &\le \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} |f(s,u(s))| ds \\ &+ \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-j-1} |f(s,u(s))| ds \\ &+ \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{i-1}}^{t_i} (t_i-s)^{q-j-1} |f(s,u(s))| ds \\ &+ \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)} |I_{j,k}(u(t_k))| + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k)+(t_k-t_i)]^j}{\Gamma(j+1)} |I_{j,i}(u(t_i))| + \sum_{j=0}^{n-1} |m_{j+1}| \\ &\le \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\ &+ \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{u_j}{\Gamma(j+1)}, \end{aligned}$$

Thus, for any $t \in J$, it follows that

$$\begin{aligned} \|u\| \le \; & \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\ &+ \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{u_j}{\Gamma(j+1)}. \end{aligned}$$

So, the set V is bounded. Therefore, by the conclusion of Theorem 2.2, the operator T has at least one fixed point. This implies that (1) has at least one solution.

Theorem 3.3. Assume that

(H_2) there exist positive constants K_i $(i = 1, 2, \dots, n+1)$ such that $|f(t,u) - f(t,v)| \leq K_1|u-v|$, $|I_{j,k}(u) - I_{j,k}(v)| \leq K_{j+2}|u-v|$, $(j = 0, 1, \dots, n-1)$ for $t \in J, u, v \in R$ and $k = 1, 2, \dots, p$. Then problem (1) has a unique solution if

$$\begin{aligned}\Lambda &= \frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\ &\quad + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)} < 1. \end{aligned} \tag{12}$$

Proof. For $u, v \in C(J)$, we have

$$\begin{aligned}|(Tu)(t) - (Tv)(t)| &\leq \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} |f(s,u(s)) - f(s,v(s))| ds \\ &\quad + \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-j-1} |f(s,u(s)) - f(s,v(s))| ds \\ &\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k) + (t_k - t_i)]^j}{\Gamma(j+1)\Gamma(q-j)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-j-1} |f(s,u(s)) - f(s,v(s))| ds \\ &\quad + \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{\Gamma(j+1)} |I_{j,k}(u(t_k)) - I_{j,k}(v(t_k))| \\ &\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{k-1} \frac{[(t-t_k) + (t_k - t_i)]^j}{\Gamma(j+1)} |I_{j,i}(u(t_i)) - I_{j,k}(v(t_k))| \\ &\leq \{\frac{L_1}{\Gamma(q+1)} + \sum_{j=0}^{n-1} \frac{L_1}{\Gamma(j+1)\Gamma(q-j+1)} + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(j+1)} \\ &\quad + \sum_{j=0}^{n-1} \frac{(p-1)L_1}{\Gamma(q-j+1)\Gamma(j+1)} + \sum_{j=0}^{n-1} \frac{L_{j+2}}{\Gamma(j+1)}\} \|u-v\| \\ &\leq \Lambda \|u-v\|. \end{aligned}$$

where Λ is given by (12). Thus, $\|Tu - Tv\| \leq \Lambda\|u-v\|$. As $\Lambda < 1$, therefore, T is a contraction operator. Hence, by the contraction mapping principle, problem (1) has a unique solution.

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